ELSEVIER

Physica D 125 (1999) 260–274

PHYSICA D

Finite element distributions in statistical theory of energy levels in quantum systems

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Received 12 December 1997; received in revised form 18 May 1998; accepted 25 August 1998
Communicated by F.H. Busse

Abstract

The probability density functions of the three-point finite elements of the three adjacent energy levels for the three-level quantum system are introduced as a supplementary characteristics of quantum chaos. The three-level quantum system is studied. The probability density functions of the second difference and asymmetrical three-point first finite element are computed for the three-dimensional Gaussian orthogonal ensemble GOE(3), the three-dimensional Gaussian unitary ensemble GUE(3), the three-dimensional Gaussian symplectic ensemble GSE(3), as well as for the Poisson ensemble PE. ©1999 Elsevier Science B.V. All rights reserved.

PACS: 05.45.+b; 02.50.+s; 24.60.Ky

Keywords: Quantum chaos; Random matrix theory; Finite elements; Distribution function

1. Introduction

In [1] we have introduced new statistical measures for description of chaos in quantum systems. The ith spacing $s_i = E_{i+1} - E_i\,(i = 1, \ldots, N-1)$ of adjacent energy levels [2–5] was interpreted as first differential quotient of energy with respect of its index. In order to take into account three-point correlations we extended s_i into first differential quotient approximated by three points:

$$\Delta^1_{a,\mathrm{fin}} E_i = \frac{1}{2(i+1-i)}(-3E_i + 4E_{i+1} - E_{i+2}), \quad i = 1, \ldots, N-2 \tag{1}$$

([1,6]; we named $\Delta^1_{a,\mathrm{fin}} E_i$ the ith asymmetrical three-point first finite element. Briefly, we call it asymmetrical element). This is three-point analog of the spacing caused by the correlation of three levels. Continuing this idea one

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0167-2789/99/$ – see front matter
PII: S0167-2789(98)00253-X

Table 1
The conditions of derivation of relevant probability density functions of finite elements from the Rosenzweig and Porter formula and equation numbers corresponding to the respective formulae.

Finite element	GOE(3), $\beta = 1$	GUE(3), $\beta = 2$	GSE(3), $\beta = 4$
$X_1 = Y_1$	Eq. (23)	Eq. (A.1)	Eq. (A.2)
$X_1 = W_1$	Eq. (24)	Eq. (A.3)	Eq. (A.4)

can introduce p-point analog of the spacing containing p-point correlations. Then, in order to describe inhomogeneity of the spectrum we introduced an analog of the second differential quotient

$$\Delta^2 E_i = \Delta^1 E_{i+1} - \Delta^1 E_i = E_i + E_{i+2} - 2E_{i+1}, \quad i = 1, \ldots, N-2, \tag{2}$$

[1]. $\Delta^2 E_i$ is spacing of the nearest neighbour spacings and it approaches zero when these neighbour spacings become equal. Thus its distribution can serve convenient measure of homogeneity. Now we see that one can create new measures by increasing the order m of the differential quotient and/or by increasing the number of points p generating wanted quotient, where $p \geq m$. Choice of a proper measure depends on what information one wants to extract from the joint probability density function of N eigenvalues of random matrix. Considered statistical measures are very easy to apply for a comparison of experimental data with theory: they do not require more work than nearest neighbour spacing measure [1]. Moreover, the finite elements are easily extendible to the study of quantum chaos in continuous spectrum, and up to our knowledge they are the only one such measures (the common Random Matrix Theory measures do not fulfill the requirement of applicability to continuous spectrum) [7].

Collecting many experimental data from the literature and comparing them to the statistical measures of $\Delta^1_{a,\mathrm{fin}} E_i$ and $\Delta^2 E_i$ we showed application of these new tools in the theory of quantum chaos.

We have demonstrated that the probability density functions of $\Delta^1_{a,\mathrm{fin}} E_i$ and $\Delta^2 E_i$ have a universal character and they properly differentiate between chaotic and integrable systems.

Purpose of the present paper is to give a mathematical basis of that statistical measure.

The paper is organized as follows: Section 2 gives a general recipe how to calculate distributions for the introduced finite elements. Then detailed calculation is performed for the second difference distribution for the three-dimensional Gaussian orthogonal ensemble GOE(3). For other cases: the three-dimensional Gaussian unitary ensemble GUE(3), and the three-dimensional Gaussian symplectic ensemble GSE(3), the final results are presented without details. The results are related to the large dimensional ensemble limit. Section 3 computes finite element distribution for the Poisson ensemble PE. Section 4 provides discussion on the homogeneity of the system.

2. The probability density functions of the finite elements for GOE(3), GUE(3), and GSE(3)

Method of derivation bases on the following procedure. The Hamiltonian matrix $H^{\mathrm{GOE}(3)}$ of the system is stochastic. Its eigenvalues E_1, E_2, E_3 are random variables (4) and the joint probability density function of the random vector (E_1, E_2, E_3) is well known (compare [8]). In order to compute the finite elements we have to sort the energies E_1, E_2, E_3 into an ascending set of new random variables $\{E_{\min}, E_{\mathrm{mes}}, E_{\max}\}$ (5). Then, we define the three-point finite element X_1, the first difference X_2, and the minimal energy X_3. We derive the cumulative distribution function of the random vector (X_1, X_2, X_3) from the joint probability density function of the random vector (E_1, E_2, E_3). Hence, we compute the cumulative distribution function of the random vector (X_1, X_2). It yields the joint probability density function of the random vector (X_1, X_2). Finally, we compute the marginal probability density functions of the three-point finite element.

A way of utilization of this method in derivation of the relevant probability density functions of the finite elements is presented in Table 1, together with the numbers of the respective formulae.

We present example of detailed calculations for the second difference $X_1 = Y_1$, for GOE(3), $\beta = 1$ (6). Results for remaining cases will be also presented, however, without details. First, let us discuss GOE(3). We consider the 3×3 Hamiltonian matrix $H^{\mathrm{GOE}(3)}$ composed of the independent random variables $H_{ij}^{\mathrm{GOE}(3)}$. The elements of $H^{\mathrm{GOE}(3)}$ are zero-centred Gaussian distributed, off-diagonal ones with the variance equal to σ^2 and diagonal ones with the variance equal to $2\sigma^2$ (compare [9–12]). For further considerations we need the joint probability density function of the random vector, composed of the eigenvalues E_i of the $H^{\mathrm{GOE}(3)}$. We obtain this from the general Rosenzweig and Porter formula [8], substituting $N = 3$ to the following

$$f_{E_1,\dots,E_N}^{\mathrm{GOE}(N)}(e_1,\dots,e_N) = C_N \prod_{\substack{i,j=1\\ i>j}}^{N} \mid e_i - e_j \mid \exp\left(-\frac{1}{4\sigma^2}\sum_{i=1}^{N} e_i^2\right), \tag{3}$$

where

$$E_i : \mathbb{R} \ni e_i \to E_i(e_i) \in \mathbb{R}, \quad i = 1,\dots,N \;\; (N = 3) \tag{4}$$

are random variables, e_1, e_2, e_3 are the sample points and $\mathbb{R}$ is a set of real numbers – sample space.

Let us define the first difference and the second one. The spacing s_1 (i.e. the first difference) is defined for two adjacent energies E_1 and E_2, that are sorted in an ascending way $E_1 \le E_2$, hence, $s_1 = E_2 - E_1$. The second difference is given for three adjacent energies E_1, E_2, and E_3 that are sorted in an ascending way $E_1 \le E_2 \le E_3$. Then, $\Delta^2 E_1 = E_1 + E_3 - 2E_2$ (compare Eq. (2)). However, the levels E_1, E_2 and E_3 calculated from GOE(3) matrix are unsorted. Therefore, we sort them in an ascending sequence of the new random variables $\{E_{\mathrm{min}}, E_{\mathrm{mes}}, E_{\mathrm{max}}\}$ as follows:

$$E_{\mathrm{min}} : \mathbb{R} \ni e_{\mathrm{min}} \to E_{\mathrm{min}}(e_{\mathrm{min}}) \in \mathbb{R},$$

$$E_{\mathrm{mes}} : \mathbb{R} \ni e_{\mathrm{mes}} \to E_{\mathrm{mes}}(e_{\mathrm{mes}}) \in \mathbb{R},$$

$$E_{\mathrm{max}} : \mathbb{R} \ni e_{\mathrm{max}} \to E_{\mathrm{max}}(e_{\mathrm{max}}) \in \mathbb{R},$$

where $e_{\mathrm{min}}, e_{\mathrm{mes}}, e_{\mathrm{max}}$ are random variables, and

$$\begin{aligned} &E_{\mathrm{max}} = \max(E_1, E_2, E_3) \text{ the maximal energy},\\ &E_{\mathrm{min}} = \min(E_1, E_2, E_3) \text{ the minimal energy},\\ &E_{\mathrm{mes}} \in \{E_1, E_2, E_3\} \backslash \{E_{\mathrm{min}}, E_{\mathrm{max}}\} \text{ the intermediate energy}. \end{aligned} \tag{5}$$

The mutual positions of the energies in the random vector (E_1, E_2, E_3) depend on the triple of the sample points (e_1, e_2, e_3). Hence, the random vector of the sorted energies $(E_{\mathrm{min}}, E_{\mathrm{mes}}, E_{\mathrm{max}})$ depends on triple of the sample points $(e_{\mathrm{min}}, e_{\mathrm{mes}}, e_{\mathrm{max}})$.

In this context we introduce three new random variables:

$$Y_1 : \mathbb{R} \ni y_1 \to Y_1(y_1) \in \mathbb{R}, \qquad Y_2 : \mathbb{R} \ni y_2 \to Y_2(y_2) \in \mathbb{R}, \qquad Y_3 : \mathbb{R} \ni y_3 \to Y_3(y_3) \in \mathbb{R},$$

defined as follows:

$$Y_1 = \Delta^2 E_{\mathrm{min}}, \qquad Y_2 = \Delta^1 E_{\mathrm{min}}, \qquad Y_3 = E_{\mathrm{min}}, \tag{6}$$

where Y_1 is the second difference of the three adjacent energies $E_{\mathrm{min}}, E_{\mathrm{mes}}, E_{\mathrm{max}}$, Y_2 is the first difference of the two adjacent energy levels $E_{\mathrm{min}}, E_{\mathrm{mes}}$, Y_3 is the lowest energy E_{min}. There are two nearest neighbour spacings for the three-level system: the "lower" one

$$s_{1\mathrm{L}} = s_{\mathrm{min}} = \Delta^1 E_{\mathrm{min}}, \tag{7}$$

and the "upper" one

$$s_{2U} = s_{mes} = \Delta^1 E_{mes}, \tag{8}$$

respectively (we have chosen the "lower" one as the Y_2).

In the first step we derive the cumulative distribution function $F_{Y_1 Y_2 Y_3}^{GOE(3)}$ of the random vector (Y_1, Y_2, Y_3) for GOE(3). The cumulative distribution function $F_{Y_1 Y_2 Y_3}^{GOE(3)}$ at the arbitrary point $(y_{10}, y_{20}, y_{30}) \in \mathbb{R}^3$ is

$$F_{Y_1 Y_2 Y_3}^{GOE(3)}(y_{10}, y_{20}, y_{30}) = P(B_{(y_{10}, y_{20}, y_{30})}), \tag{9}$$

where

$$B_{(y_{10}, y_{20}, y_{30})} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : Y_1(y_1) \leq y_{10}, Y_2(y_2) \leq y_{20}, Y_3(y_3) \leq y_{30}\}, \tag{10}$$

[13]. It follows that

$$P(B_{(y_{10}, y_{20}, y_{30})}) = \int_{B_{(y_{10}, y_{20}, y_{30})}} dy_1 dy_2 dy_3 \, f_{Y_1 Y_2 Y_3}^{GOE(3)}(y_1, y_2, y_3), \tag{11}$$

where $f_{Y_1 Y_2 Y_3}^{GOE(3)}$ is the joint probability density function of the random vector (Y_1, Y_2, Y_3) for GOE(3). From the definition of the random vector (6) we deduce that it always holds:

$$Y_1(y_1) + Y_2(y_2) \geq 0, \quad Y_2(y_2) \geq 0. \tag{12}$$

Therefore, collecting (9)–(12), one has:

$$F_{Y_1 Y_2 Y_3}^{GOE(3)}(y_{10}, y_{20}, y_{30}) = \Theta(y_{10} + y_{20})\Theta(y_{20}) \int_{B_{(y_{10}, y_{20}, y_{30})}} dy_1 dy_2 dy_3 \, f_{Y_1 Y_2 Y_3}^{GOE(3)}(y_1, y_2, y_3), \tag{13}$$

where Θ is the Heaviside function. There does not exist local homeomorphism of class C^1 on $\mathbb{R}^3$ between random vectors (E_1, E_2, E_3) and (Y_1, Y_2, Y_3). However, if we divide $\mathbb{R}^3$ into six separate sets A_i, so that $\cup_{i=1}^{6} A_i = \mathbb{R}^3$, $A_i \cap A_j = \phi$, for $i \neq j$, and

$$A_i = \{(e_1, e_2, e_3) \in \mathbb{R}^3 : E_{\pi_i(1)}(e_1) \geq E_{\pi_i(2)}(e_2) \geq E_{\pi_i(3)}(e_3)\} \quad \text{for } i = 1, \ldots, 6, \tag{14}$$

where π_i is the ith element of the permutation group S_3 of the set $I_3 = \{1, 2, 3\}$; then we obtain six local homeomorphisms ξ_i of class C^1 on each A_i. Namely,

$$\xi_i : A_i \ni (e_1, e_2, e_3) \to (e_{\pi_i(1)}, e_{\pi_i(2)}, e_{\pi_i(3)}) \in \mathbb{R}^3 \quad \text{for } i = 1, \ldots, 6. \tag{15}$$

Hence, from (13)–(15), and change of variable formula [14], one has

$$F_{Y_1 Y_2 Y_3}^{GOE(3)}(y_{10}, y_{20}, y_{30}) = \Theta(y_{10} + y_{20})\Theta(y_{20}) \sum_{\pi_i \in S_3} \int_{\xi_i^{-1}(B_{(y_{10}, y_{20}, y_{30})})} de_1 de_2 de_3 \, f_{E_1 E_2 E_3}^{GOE(3)}(e_1, e_2, e_3). \tag{16}$$

We use again the change of variable formula

$$F_{Y_1 Y_2 Y_3}^{GOE(3)}(y_{10}, y_{20}, y_{30}) = \Theta(y_{10} + y_{20})\Theta(y_{20}) \sum_{\pi_i \in S_3} \int_{B_{(y_{10}, y_{20}, y_{30})}} dy_1^i dy_2^i dy_3^i \, f_{\xi^i(E_1, E_2, E_3)}^{GOE(3)}(y_1^i, y_2^i, y_3^i). \tag{17}$$

The integrals in (17) are equal to each other, hence

$$F_{Y_1 Y_2 Y_3}^{GOE(3)}(y_{10}, y_{20}, y_{30}) = 6\Theta(y_{10} + y_{20})\Theta(y_{20}) \int_{B_{(y_{10}, y_{20}, y_{30})}} dy_1^1 dy_2^1 dy_3^1 \, f_{\xi^1(E_1, E_2, E_3)}^{GOE(3)}(y_1^1, y_2^1, y_3^1). \tag{18}$$

Finally, one easily infers from (3) applied for $N = 3$, (15) and (18):

$$F^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3}(y_{10}, y_{20}, y_{30}) = 6\Theta(y_{10}+y_{20})\Theta(y_{20}) \sum_{\pi_i \in S_3} \int_0^{y_{20}} \mathrm{d}y_2^1 \int_{-y_2^1}^{y_{10}} \mathrm{d}y_1^1 \int_{-\infty}^{y_{30}} \mathrm{d}y_3^1 \, a_{\pi_1}(y_1^1, y_2^1, y_3^1) \times \exp\left(-\frac{1}{4\sigma^2} b_{\pi_1}(y_1^1, y_2^1, y_3^1)\right), \tag{19}$$

where the "amplitude" function is

$$a_{\pi_1}(y_1^1, y_2^1, y_3^1) = C_3 \mid y_1^1 + y_2^1 \mid\mid y_2^1 \mid\mid y_1^1 + 2y_2^1 \mid,$$

and the "exponent" function is

$$b_{\pi_1}(y_1^1, y_2^1, y_3^1) = (y_1^1 + 2y_2^1 + y_3^1)^2 + (y_2^1 + y_3^1)^2 + (y_3^1)^2,$$

respectively (for more details see [15]).

Now we can compute the marginal distribution function $F^{\mathrm{GOE(3)}}_{Y_1Y_2}$ of the random vector (Y_1, Y_2) for GOE(3). It is necessary to determine the support of the joint probability density function of the random vector (Y_1, Y_2, Y_3). The support of a function $f : D \ni x \to f(x) \in R$ is a closure of such domain subset in which function is not equal to zero

$$\mathrm{supp}\, f = \mathrm{Cl}\{x \in D : f(x) \neq 0\}.$$

Then,

$$\int_D \mathrm{d}x\, f(x) = \int_{\mathrm{supp}\, f} \mathrm{d}x\, f(x).$$

This property will be used in the further calculations.

We see that the support of the cumulative distribution function $F^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3}$ and the support of the joint probability density function $f^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3}$ are equal to the set A

$$\mathrm{supp}\, f^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3} = \mathrm{supp}\, F^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3} = A = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : Y_1(y_1) + Y_2(y_2) \geq 0 \text{ and } Y_2(y_2) \geq 0\}.$$

The properties of the cumulative distribution function yield to (compare [13])

$$F^{\mathrm{GOE(3)}}_{Y_1Y_2}(y_{10}, y_{20}) = F^{\mathrm{GOE(3)}}_{Y_1Y_2Y_3}(y_{10}, y_{20}, \infty) = 6\Theta(y_{10}+y_{20})\Theta(y_{20}) \int_0^{y_{20}} \mathrm{d}y_2 \int_{-y_2}^{y_{10}} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{d}y_3 \, a_{\pi_1}(y_1, y_2, y_3) \exp\left(-\frac{1}{4\sigma^2} b_{\pi_1}(y_1, y_2, y_3)\right).$$

Finally,

$$F^{\mathrm{GOE(3)}}_{Y_1Y_2}(y_{10}, y_{20}) = \Theta(y_{10}+y_{20})\Theta(y_{20}) C_3 \int_0^{y_{20}} \mathrm{d}y_2 \times \int_{-y_2}^{y_{10}} \mathrm{d}y_1 \, 4\sqrt{3\pi}\sigma(y_1+y_2)y_2(y_1+2y_2) \exp\left(-\frac{y_1^2 + 3y_1y_2 + 3y_2^2}{6\sigma^2}\right). \tag{20}$$

We see that the support of the marginal distribution function $F^{\mathrm{GOE(3)}}_{Y_1Y_2}$ is equal to the set B

$$\mathrm{supp}\, F^{\mathrm{GOE(3)}}_{Y_1Y_2} = B = \{(y_1, y_2) \in \mathbb{R}^3 : Y_1(y_1) + Y_2(y_2) \geq 0 \text{ and } Y_2(y_2) \geq 0\}.$$

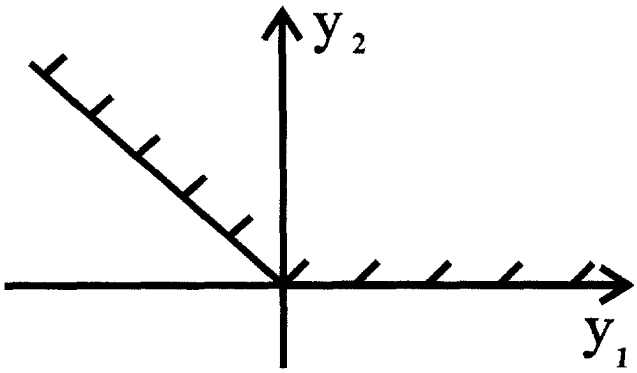

Fig. 1. The support of the functions $F_{Y_1Y_2}^{\mathrm{GOE}(3)}$ and $f_{Y_1Y_2}^{\mathrm{GOE}(3)}$.

From (20) we compute the joint probability density function $f_{Y_1Y_2}^{\mathrm{GOE}(3)}$ of the random vector (Y_1, Y_2) for GOE(3). Its definition yields to

$$f_{Y_1Y_2}^{\mathrm{GOE}(3)}(y_1, y_2) = \frac{\partial^2 F_{Y_1Y_2}^{\mathrm{GOE}(3)}}{\partial y_1 \partial y_2}(y_1, y_2).$$

Therefore,

$$f_{Y_1Y_2}^{\mathrm{GOE}(3)}(y_1, y_2) = \Theta(y_1 + y_2)\Theta(y_2)C_3 4\sqrt{3\pi}\sigma(y_1 + y_2)y_2(y_1 + 2y_2)\exp\left(-\frac{y_1^2 + 3y_1y_2 + 3y_2^2}{6\sigma^2}\right). \quad (21)$$

Hence,

$$\operatorname{supp} f_{Y_1Y_2}^{\mathrm{GOE}(3)} = \operatorname{supp} F_{Y_1Y_2}^{\mathrm{GOE}(3)} = B.$$

Now we can separately compute the probability density functions of the first and the second differences. We plot the support of the function $f_{Y_1Y_2}^{\mathrm{GOE}(3)}$ in the coordinates (y_1, y_2) to facilitate further computations (see Fig. 1).

Let us compute the probability density function of the second difference $f_{Y_1}^{\mathrm{GOE}(3)}$ for GOE(3). From (21) and from Fig. 1 one can notice that

$$f_{Y_1}^{\mathrm{GOE}(3)}(y_1) = \int_0^{\infty} \mathrm{d}y_2\, f_{Y_1Y_2}^{\mathrm{GOE}(3)}(y_1, y_2) \quad \text{for } y_1 \geq 0,$$

$$f_{Y_1}^{\mathrm{GOE}(3)}(y_1) = \int_{-y_1}^{\infty} \mathrm{d}y_2\, f_{Y_1Y_2}^{\mathrm{GOE}(3)}(y_1, y_2) \quad \text{for } y_1 < 0. \quad (22)$$

Hence,

$$f_{Y_1}^{\mathrm{GOE}(3)}(y_1) = C_3 2^4 \sqrt{3\pi}\sigma^5 \exp\left(-\frac{y_1^2}{6\sigma^2}\right) \quad (23)$$

for any y_1. The coefficient C_3 is determined from the normalization condition and it reads

$$C_3 = \frac{1}{\int_{-\infty}^{\infty} \mathrm{d}y_1\, f_{Y_1}^{\mathrm{GOE}(3)}(y_1)} = \frac{\sqrt{2}}{2^5 \cdot 3\pi\sigma^6}.$$

The way of comparing the finite element distributions for GOE(3), GUE(3), GSE(3), and for the Poisson ensemble with each other is to make the finite element dimensionless. This can be achieved by dividing them by the mean value

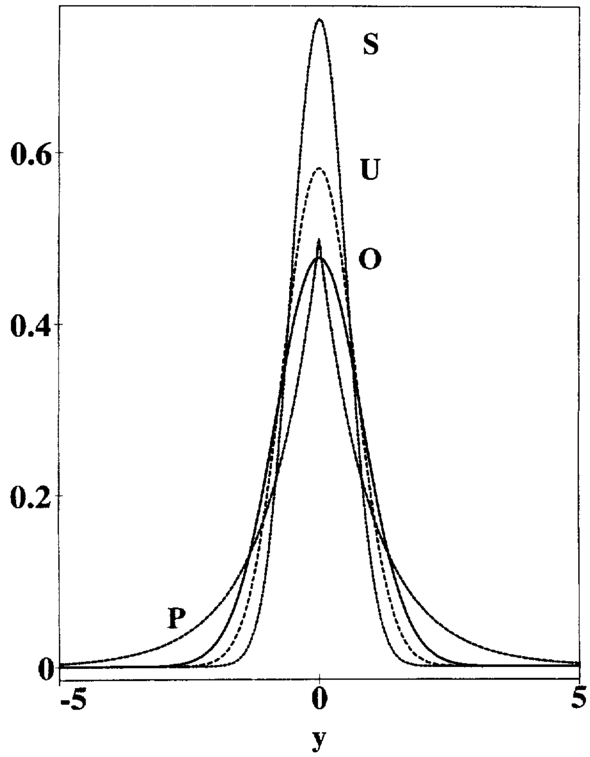

Fig. 2. The probability density function of the dimensionless second difference for the Poisson ensemble (P: medium dashed line), for GOE(3) (O: solid line), for GUE(3) (U: medium dashed line), and for GSE(3) (S: short dashed line). The value of y is the ratio of second difference to the mean spacing for GOE(3), GUE(3), GSE(3), and Poisson ensemble, respectively.

of the spacing for GOE(3), GUE(3), GSE(3), and PE, respectively. We present the plot of the dimensionless second difference for GOE(3), GUE(3), GSE(3), and PE in Fig. 2. We should point out that the result random variable Y_1 is a zero-centred Gaussian distributed but with the variance that differs from the variances of the Hamiltonian matrix elements σ^2, $2\sigma^2$ (compare Eq. (23)).

Below we present the results for other cases listed in Table 1.

The probability density functions of the second difference for GUE(3) and GSE(3) are given in Appendix A. Results (23), (A.1) and (A.2), for GOE(3), GUE(3), and GSE(3), respectively, have different analytical forms.

Using the method presented in this section with the appropriate support [15] we derive the probability density functions of the asymmetrical element $W_1 = \Delta^1_{a,\mathrm{fin}} E_{\min}$ for GOE(3):

$$f_{W_1}^{\mathrm{GOE}(3)}(w_1) = A_3[B_3(1-a_1) + D_3 a_4]a_2 \quad \text{for } w_1 \geq 0,$$

$$f_{W_1}^{\mathrm{GOE}(3)}(w_1) = A_3[B_3(1+a_3) + D_4 a_5]a_2 \quad \text{for } w_1 \leq 0, \tag{24}$$

where

$$a_1 = \mathrm{erf}\left(\frac{5\sqrt{78}w_1}{234\sigma}\right), \qquad a_2 = \exp\left(-\frac{w_1^2}{26\sigma^2}\right), \qquad a_3 = \mathrm{erf}\left(\frac{7w_1}{\sqrt{78}\sigma}\right), \qquad a_4 = \exp\left(-\frac{25w_1^2}{702\sigma^2}\right),$$

$$a_5 = \exp\left(-\frac{49w_1^2}{78\sigma^2}\right),$$

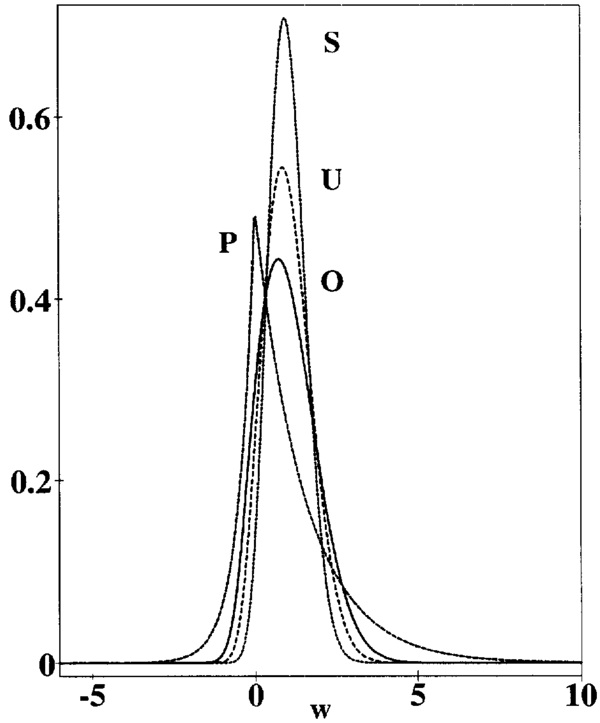

Fig. 3. The probability density function of the dimensionless asymmetrical element for the Poisson ensemble (P: medium dashed line), for GOE(3) (O: solid line), or GUE(3) (U: medium dashed line), and for GSE(3) (S: short dashed line). The value of w is the ratio of asymmetrical element to the mean spacing for GOE(3), GUE(3), GSE(3), and Poisson ensemble, respectively.

Table 2
Power law exponents for spacing distributions.

	ensemble					
	GOE(2), $\beta = 1$	GOE(3), $\beta = 1$	GUE(2), $\beta = 2$	GUE(3), $\beta = 2$	GSE(2), $\beta = 4$	GSE(3), $\beta = 4$
p	1	1	2	2	4	4

$$A_3 = C_3 \frac{2^3}{13^4} \sqrt{3\pi}\sigma^2, \qquad B_3 = 5 \cdot 7\sqrt{78\pi} w_1(-w_1^2 + 3 \cdot 13\sigma^2),$$
$$D_3 = 2 \cdot 3^2 \cdot 13\sigma(7w_1^2 + 2^3 \cdot 3 \cdot 13\sigma^2), \qquad D_4 = 2 \cdot 3 \cdot 13\sigma(-5w_1^2 + 2^2 \cdot 3^2 \cdot 13\sigma^2),$$

and $\mathrm{erf}(x)$ is the error function.

The probability density functions of the asymmetrical element W_1 for GUE(3) and GSE(3) are given in Appendix A.

We present the plot of the dimensionless asymmetrical element distributions for GOE(3), GUE(3), GSE(3), and PE in Fig. 3.

Asymptotic behaviour around origin of the spacing distributions are characterized by the power law exponents

$$p = \lim_{w \to 0^+} \frac{\ln f_W^{\mathrm{ENS}}(w)}{\ln w}, \tag{25}$$

where ENS = GOE, GUE, GSE. Table 2 presents values of p for the found distributions [15]. From Table 2 we read

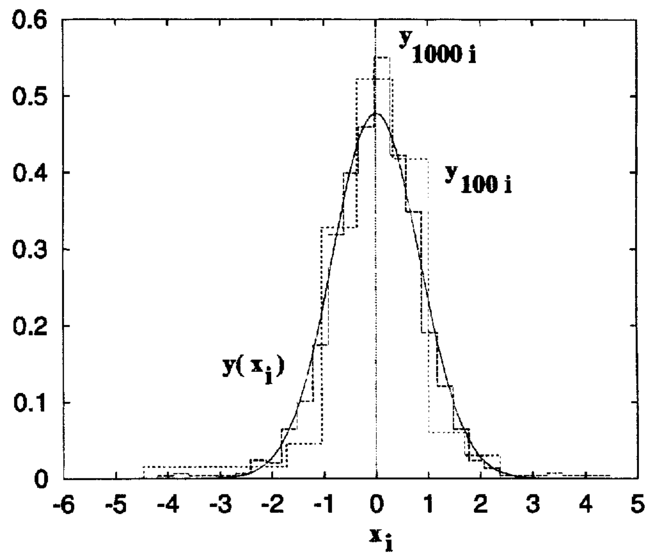

Fig. 4. The marginal probability density function of the second difference Y_1 for GOE(3) $y(x_i)$ (solid line), for GOE(100) y_{100_i} (histogram: thick short dashed stepwise line), and for GOE(1000) y_{1000_i} (histogram: thin long dashed stepwise line).

that characteristic exponent p is equal to β parameter from the Rosenzweig and Porter formula (3). Conformity of p and β corroborates our suggestion on relation between spacing and first differential quotient [7].

We relate our results to the large dimensional ensemble limit. The probability density function of the second difference for GOE(3) approximates the probability density function of the second difference for infinite dimensional GOE. We have performed computer simulations of GOE(N), where $N = 100, 200, 300, 400$, and $N = 1000$. We have calculated the spectrum of N eigenvalues of the random matrix and we unfolded it using the standard procedure. We present the results in Fig. 4, plotting the exact rescaled probability density function y of the second difference for GOE(3), and the computer simulations of this probability density function for GOE(100), and for GOE(1000). We rescale the variable on the horizontal axis x_i (x_i is equal to the quotient of the second difference and mean spacing D), hence, $y(x_i) = D f_{Y_1}^{\mathrm{GOE}(3)}(x_i D)$. From Fig. 4 one easily sees that the probability density function of the second difference for GOE(3) approximates the probability density function of the second difference for GOE($N \to \infty$) excellently. This observation has been confirmed by the calculation of the mean square deviation

$$S_N = \sqrt{\frac{\sum_{i=1}^{n_N} (y_{N_i} - y(x_i))^2}{n_N - 1}}, \tag{26}$$

where y_{N_i} is the value of the histogram of the probability density function of the second difference for GOE(N) at the ith bin x_i, whereas $y(x_i)$ is computed from the analytical formula for $f_{Y_1}^{\mathrm{GOE}(3)}$ (S_N is legal measure of error, because the fluctuations $y_{N_i} - y(x_i)$ are Gaussian distributed). The sequence of S_N is not monotone, it is asymptotically decreasing to a positive small number (never approaching zero: $S_{100} = 0.213 \times 10^{-2}$, $S_{200} = 0.266 \times 10^{-2}$, $S_{300} = 0.193 \times 10^{-2}$, $S_{400} = 0.207 \times 10^{-2}$, $S_{1000} = 0.189 \times 10^{-2}$). Analogously, the investigations conducted for asymmetrical element confirm the utility of formula (24) to the analysis of random matrices of high dimension [15].

3. The probability density functions of the finite elements for the Poisson ensemble

We assume, following [10], that the energy levels are uncorrelated and randomly distributed for the Poisson ensemble. If the average spacing of the level energies equals D, then the spacing distribution is the Poisson one

$$P_P(s) = \frac{1}{D}\exp\left(-\frac{s}{D}\right) \quad \text{for } s \geq 0, \qquad P_P(s) = 0 \quad \text{for } s < 0. \tag{27}$$

We will compute the probability density function of the second difference $f_{\Delta^2 E_1^I,L}$ for the Poisson ensemble (where index L is used to facilitate notation). This function is the natural extension of the Poisson distribution Eq. (27) and closes the investigation on the three energy level functions for the quantum systems that belong to GOE(3), GUE(3), GSE(3), or PE. We assume that we study three adjacent energy levels and we introduce the three corresponding random variables E_1^I, E_2^I, E_3^I that are the energies of the three levels. We assume that

$$E_i^I : \mathbb{R} \ni e_i^I \to E_i^I(e_i^I) \in \mathbb{R} \quad \text{for } i = 1, \ldots, 3, \qquad E_1^I(e_1^I) \leq E_2^I(e_2^I) \leq E_3^I(e_3^I) \quad \text{for } (e_1^I, e_2^I, e_3^I) \in \mathbb{R}^3.$$

First, we compute the joint probability density function of the random vector (Y_1^I, Y_2^I), where

$$Y_1^I = \Delta^2 E_1^I, \qquad Y_2^I = \Delta^1 E_1^I,$$
$$Y_i^I : \mathbb{R} \ni y_i^I \to Y_i^I(y_i^I) \in \mathbb{R} \quad \text{for } i = 1, 2.$$

We assume that

$$E_1^I(e_1^I) = \epsilon_1, \quad E_2^I(e_2^I) = \epsilon_2, \quad E_3^I(e_3^I) = \epsilon_3.$$

We introduce new real variables $t = E_1^I(e_1^I) + E_3^I(e_3^I) - 2E_2^I(e_2^I)$ and $s = E_2^I(e_2^I) - E_1^I(e_1^I)$. Thus, $E_3^I(e_3^I) - E_2^I(e_2^I) = t + s$. The probability that the value of the second energy is not contained in the interval $[\epsilon_1, \epsilon_1 + s]$ is

$$P(\{e_2^I : E_2^I(e_2^I) \geq \epsilon_1 + s\}) = \int_s^\infty \mathrm{d}s' P_P(s'),$$

where P_P is the Poisson distribution (27). The conditional probability to find the value of the second energy in the interval $[\epsilon_1 + s, \epsilon_1 + s + \mathrm{d}s]$, while the value of the first energy equals ϵ_1, is

$$P(\{e_2^I : \epsilon_1 + s \leq E_2^I(e_2^I) \leq \epsilon_1 + s + \mathrm{d}s\}) = \frac{1}{D}.$$

The probability that the value of the third energy is not contained in the interval $[\epsilon_2, \epsilon_2 + s + t]$ is

$$P(\{e_3^I : E_3^I(e_3^I) \geq \epsilon_2 + s + t\}) = \int_{s+t}^\infty \mathrm{d}s'' P_P(s'').$$

The conditional probability to find the value of the third energy in the interval $[\epsilon_2 + s + t, \epsilon_2 + s + t + \mathrm{d}t]$, while the value of the second energy equals $\epsilon_1 + s$, is

$$P(\{e_3^I : \epsilon_2 + s + t \leq E_3^I(e_3^I) \leq \epsilon_2 + s + t + \mathrm{d}t\}) = \frac{1}{D}.$$

Hence, the joint probability density function of the random vector (Y_1^I, Y_2^I) is

$$f_{Y_1^I Y_2^I}(t, s) = \int_s^\infty \mathrm{d}s' P_P(s') \frac{1}{D} \int_{s+t}^\infty \mathrm{d}s'' P_P(s'') \frac{1}{D} = \frac{1}{D^2}\exp\left(-\frac{t + 2s}{D}\right). \tag{28}$$

Therefore, the marginal probability density function of the second difference is

$$f_{Y_1^I,L}(t) = \int_0^\infty \mathrm{d}s\, f_{Y_1^I Y_2^I}(t, s) = \frac{1}{2D}\exp\left(-\frac{t}{D}\right) \qquad \text{for } t \geq 0,$$

because the value of the first difference Y_2^I is an arbitrary nonnegative real number s for $t \geq 0$ and thus, one can integrate $f_{Y_1^I Y_2^I}$ over $s \geq 0$ (we again use the method presented in Section 2). Up till now we have assumed that the second difference's value is nonnegative. If we repeat our reasoning also for negative t, then we finally obtain

$$f_{Y_1^I,L}(t) = \frac{1}{2D} \exp\left(-\frac{|t|}{D}\right), \tag{29}$$

which is the Laplace distribution. One must agree that the probability density function of the second difference has to be symmetric in the variable t, because the positive and negative second differences are equally probable.

One can analogously derive the probability density function of the asymmetrical element $W_1^I = \Delta^1_{a,\mathrm{fin}} E_1^I$

$$f_{W_1^I}(w_1) = \frac{1}{2D} \exp\left(-\frac{2w_1}{3D}\right) \quad \text{for } w_1 \geq 0, \quad f_{W_1^I}(w_1) = \frac{1}{2D} \exp\left(\frac{2w_1}{D}\right) \quad \text{for } w_1 < 0. \tag{30}$$

One can see that the function $f_{W_1^I}$ is asymmetrical and it has the maximum at the origin. The function $f_{W_1^I}$ is of class $C^0(\mathbb{R}) \backslash C^1(\mathbb{R})$. The first derivative of $f_{W_1^I}$ is discontinuous at the origin and the jump is

$$\lim_{w_1 \to 0^+} \frac{\mathrm{d} f_{W_1^I}(w_1)}{\mathrm{d} w_1} - \lim_{w_1 \to 0^-} \frac{\mathrm{d} f_{W_1^I}(w_1)}{\mathrm{d} w_1} = -\frac{4}{3D^2}. \tag{31}$$

4. Conclusions

Statistics of the second difference $\Delta^2 E_1$ distinguishes between the Poisson ensemble and GOE(3). The Poisson ensemble is characterized by the Laplace distribution, whereas GOE(3) is characterized by the Gaussian one. However, these distributions look quite similar, because both have the maximum at $\Delta^2 E_1 = 0$. In order to distinguish between the Laplace distribution and the Gaussian one, we calculate their first derivatives at the origin. The distribution function characterizing Poisson ensemble is singular at $\Delta^2 E_1 = 0$, its derivative does not exist and it exhibits a characteristic jump

$$\lim_{t \to 0^+} \frac{\mathrm{d} f_{Y_1^I,L}(t)}{\mathrm{d} t} - \lim_{t \to 0^-} \frac{\mathrm{d} f_{Y_1^I,L}(t)}{\mathrm{d} t} = -\frac{1}{D^2}. \tag{32}$$

On the other hand, the first derivative of the distribution function characterizing GOE(3), GUE(3), GSE(3), is continuous in the whole domain and it is equal to zero at the origin

$$\frac{\mathrm{d} f_{Y_1}^{\mathrm{GOE}(3)}}{\mathrm{d} y_1}(0) = \frac{\mathrm{d} f_{Y_1}^{\mathrm{GUE}(3)}}{\mathrm{d} y_1}(0) = \frac{\mathrm{d} f_{Y_1}^{\mathrm{GSE}(3)}}{\mathrm{d} y_1}(0) = 0. \tag{33}$$

The spacing distributions: Wigner's distribution [2–5], and Poisson's distribution (27) provide information only about the interaction between the two adjacent energy levels. However, the second difference distributions (23), (A.1) and (A.2) provide also supplementary information about the three adjacent energy levels. For the three levels one can talk about the homogeneity of the level spacing (equal distance three-level distribution).

It is interesting to discuss the relation between $\Delta^1_{a,\mathrm{fin}} E_i$, $\Delta^2 E_i$, and three-point correlation functions introduced earlier by Bohigas et al., [16,17]. The distribution of the spacing can be derived from the two-level correlation function for GOE(2), GUE(2), and GSE(2), whereas the distributions of the second difference and asymmetrical element can be derived from the three-level correlation function for GOE(3), GUE(3), and GSE(3), [18–20]. The asymmetrical element is a three-level derivative magnitude of the spacing approximated by three points. It is not a three-level fluctuation measure, unlike the skewness γ_1 (see [17], Eq. (5)). However, the distribution of asymmetrical element, unlike the spacing distribution, measures three-point level repulsion. The spacing and the

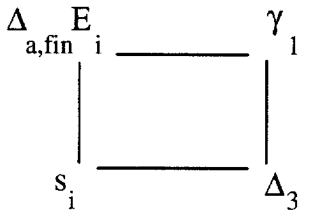

Fig. 5. The relations between asymmetrical element $\Delta^1_{a,\mathrm{fin}} E_i$, skewness γ_1, spacing s_i, and Δ_3 statistics.

two-level fluctuation measures, for example spectral averaged Δ_3 statistics [16] or number variance statistics $\Sigma^2(\bar{n})$ [21], are derived from two-level correlation functions. Since the skewness has been derived from the three-level correlation function, the asymmetrical element is a missing three-level quantity that supplements the two-level and the three-level framework, and it has the connections both with the spacing and with the skewness. The above-mentioned relations between the spacing, Δ_3, the asymmetrical element, and the skewness are summarized in the following diagram (compare Fig. 5).

In the upper horizontal line one has three-level quantities describing the level repulsion of the system and three-level fluctuations of spectrum, respectively, whereas in the lower horizontal line one has two-level quantities describing the level repulsion and two-level fluctuations of spectrum, respectively. The vertical axis is connected with number p of the points approximating first differential quotient, whereas the horizontal one gives two classes: level repulsion measures or fluctuation measures, respectively.

The second difference does not have an analog measure of both homogeneity and fluctuations of the spectrum.

Now we discuss how the second difference $\Delta^2 E_1$ describes some details of the distribution of the three energy levels.

First, we study the Gaussian ensembles and we assume that every two adjacent levels repel each other. We sort the levels in an ascending sequence $\{E_1, E_2, E_3\}$. If E_1, E_2 repel each other and also E_2, E_3 repel each other, then the three-level system tends to be homogeneous. The homogeneous system is the state for which the "lower" spacing and the "upper" one are equal to each other

$$s_{1\mathrm{L}} = E_2 - E_1 = E_3 - E_2 = s_{2\mathrm{U}}.$$

The levels E_1 and E_3 tend to be remote from each other. The intermediate level E_2 tends to be in the equal distance between them. Therefore,

$$\Delta^2 E_1 = s_{2\mathrm{U}} - s_{1\mathrm{L}} = 0,$$

and the population of the probability density function of the second difference has a maximum at $\Delta^2 E_1 = 0$. If the "lower" spacing and the "upper" one are not equal to each other, then the second difference is not equal to zero. If we assume that $s_{2\mathrm{U}} > s_{1\mathrm{L}}$, then $\Delta^2 E_1 > 0$ and the system is inhomogeneous. Hence, the probability density function of the second difference for $\Delta^2 E_1 > 0$ does not reach the maximum and such situation is less probable. The probability density functions of the second difference (23), (29), (A.1) and (A.2), describe properly the above-mentioned situation. Up till now we have assumed that the three-level system has the property that every two adjacent levels repel each other. It means that the system might be treated as two-level one belonging to the Gaussian ensemble from the point of view of the nearest neighbour level spacing distribution.

Second, let us assume that the quantum three-level system is described by Poisson ensemble. Hence, the levels are uncorrelated and randomly distributed and the spacing of the two adjacent levels is governed by the Poisson distribution (27). The second difference of the three adjacent levels is governed by the Laplace distribution (29). We can say that the system tends to be homogeneous, because the system with $\Delta^2 E_1 = 0$ is the most probable whereas the one with $\Delta^2 E_1 \neq 0$ is less probable (compare Eq. (29)). Again the second difference distribution discerns between the homogeneous system and the inhomogeneous one.

The second difference and the "lower" spacing are anticorrelated for GOE(3), GUE(3), GSE(3), and PE, [15]. The linear independence of the second difference and the "lower" spacing is expressed by the following correlation coefficient:

$$\rho(Y_1, Y_2) = -\frac{\sqrt{6}\pi}{2\sqrt{11\pi^2 - 27\pi}} \approx -0.790 \quad \text{for GOE(3)}, \tag{34}$$

since the absolute value of $\rho(Y_1, Y_2)$ is not equal to 1. Therefore, the distributions of the second difference carry new information not contained in the distributions of the spacing. The analogous calculations prove that the asymmetrical element and the "lower" spacing are correlated and linearly independent random variables for GOE(3), GUE(3), GSE(3), and PE [15]. Hence, the probability density functions of the asymmetrical element include information that cannot be extracted from the ones of spacing. The above discussion also motivates the study of the second difference and asymmetrical element.

The second differential quotient should be compared to the curvature of energy levels introduced by Delande and Zakrzewski [22] and Zakrzewski et al. [23]. The "motion" of the levels with respect to the "fictitious" time λ is studied. Namely, the Hamiltonian operator of the quantum system H linearly depends on the *continuous* parameter λ

$$H(\lambda) = H_1 + \lambda H_2, \tag{35}$$

where H_1 and H_2 are the operators describing some parts of the dynamics. Hence, the second derivative of the energy level with respect to λ is introduced

$$K = \frac{\mathrm{d}^2 E(i)}{\mathrm{d}\lambda^2}. \tag{36}$$

The second derivative K is called the "curvature" of levels. Comparing (2) with (36) one easily sees that the second difference is the discrete analogy of the continuous curvature in "perpendicular" direction to λ. Hence, we can treat the second difference as the curvature of the level E_i with respect to its discrete label parameter i. We point out some differences in both approaches. The change of the parameter λ causes the change of the system, but the change of the parameter i causes the change of the energy in the same system. This means that the study of the distribution of the curvature with respect to the parameter λ is a tool for comparing the different and close to each other systems. Whereas the study of the system by the distribution of the second difference allows to investigate one system without perturbation, this approach introduces other information about the system than K. Therefore two approaches: the use of the distribution of second difference $\Delta^2 E_i$ and of the distribution of curvature K are not equivalent to each other. Combination of these methods together makes the investigation space "two-dimensional": one can study the changes of the function f with respect to i and to λ ("space" and "time" coordinates).

We have used the Hewlett Packard work station, model 715/33, to the symbolic computations. The following results: (20), (21), (23), (24) and (A.1)–(A.4), have been derived by the Maple V program, release 3.

Appendix A

The probability density functions of the second difference for GUE(3) and GSE(3) are

$$f_{Y_1}^{\mathrm{GUE(3)}}(y_1) = A_1[B_1(1-a) + D_1 b]c \quad \text{for } y_1 \geq 0,$$
$$f_{Y_1}^{\mathrm{GUE(3)}}(y_1) = f_{Y_1}^{\mathrm{GUE(3)}}(-y_1) \quad \text{for } y_1 \leq 0, \tag{A.1}$$

and

$$f_{Y_1}^{\mathrm{GSE(3)}}(y_1) = A_2[B_2(1-a) + D_2 b]c \quad \text{for } y_1 \geq 0,$$
$$f_{Y_1}^{\mathrm{GSE(3)}}(y_1) = f_{Y_1}^{\mathrm{GSE(3)}}(-y_1) \quad \text{for } y_1 \leq 0, \tag{A.2}$$

where

$$a = \mathrm{erf}\left(\frac{\sqrt{2}y_1}{4\sigma}\right), \qquad b = \exp\left(-\frac{y_1^2}{8\sigma^2}\right), \qquad c = \exp\left(-\frac{y_1^2}{24\sigma^2}\right),$$

$$A_1 = \frac{\sqrt{3}}{2^9 \cdot 3\pi\sigma^5}, \qquad B_1 = \sqrt{2\pi}(y_1^4 - 2^3 \cdot 3\sigma^2 y_1^2 + 2^4 \cdot 3 \cdot 5\sigma^4),$$

$$D_1 = 2^2\sigma y_1(-y_1^2 + 2^2 \cdot 3 \cdot 5\sigma^2), \qquad A_2 = \frac{\sqrt{3}}{2^{17} \cdot 3^5 \cdot 5\pi\sigma^9},$$

$$B_2 = \sqrt{2\pi}(y_1^8 - 2^4 5\sigma^2 y_1^6 + 2^5 3 \cdot 5 \cdot 7\sigma^4 y_1^4 - 2^8 3^2 \cdot 5 \cdot 7\sigma^6 y_1^2 + 2^8 3^2 \cdot 5 \cdot 7 \cdot 11\sigma^8),$$

$$D_2 = \sigma y_1(-2^2 y_1^6 + 2^4 3 \cdot 7\sigma^2 y_1^4 - 2^6 3 \cdot 5 \cdot 7\sigma^4 y_1^2 + 2^8 3^2 \cdot 5 \cdot 7 \cdot 11\sigma^6).$$

The probability density functions of the asymmetrical element W_1 for GUE(3) and GSE(3) are

$$\begin{aligned} f_{W_1}^{\mathrm{GUE}(3)}(w_1) &= C_{3,2}[B_5 a_2(1 - a_1) + D_5 a_9] \quad \text{for } w_1 \geq 0, \\ f_{W_1}^{\mathrm{GUE}(3)}(w_1) &= C_{3,2}[B_5 a_2(1 + a_3) + D_6 a_{10}] \quad \text{for } w_1 \leq 0, \end{aligned} \tag{A.3}$$

where

$$C_{3,2} = \frac{1}{2^8 \cdot 3\pi^{3/2}\sigma^9}, \qquad a_9 = \exp\left(-\frac{2w_1^2}{27\sigma^2}\right), \qquad a_{10} = \exp\left(-\frac{2w_1^2}{3\sigma^2}\right),$$

$$B_5 = \frac{2^4 \cdot 3\sqrt{26}\pi\sigma^2}{13^7}(5^2 \cdot 7^2 w_1^6 + 2 \cdot 3 \cdot 13 \cdot 233\sigma^2 w_1^4 + 3^3 \cdot 13^2 \cdot 577\sigma^4 w_1^2 + 2^2 \cdot 3^6 \cdot 5 \cdot 13^3 \sigma^6),$$

$$D_5 = \frac{2^5 \cdot 3^2 \cdot 5\sqrt{3\pi}\sigma^3}{13^6} w_1(-7^2 w_1^4 - 3 \cdot 13\sigma^2 w_1^2 + 2^2 \cdot 3^2 \cdot 13^2 \cdot 71\sigma^4),$$

$$D_6 = \frac{2^5 \cdot 3 \cdot 7\sqrt{3\pi}\sigma^3}{13^6} w_1(5^2 w_1^4 + 3^3 \cdot 13\sigma^2 w_1^2 - 2^2 \cdot 3^2 \cdot 5 \cdot 13^2\sigma^4),$$

and

$$\begin{aligned} f_{W_1}^{\mathrm{GSE}(3)}(w_1) &= A_7[B_7(1 - a_1) + D_7 a_4]a_2 \quad \text{for } w_1 \geq 0, \\ f_{W_1}^{\mathrm{GSE}(3)}(w_1) &= A_7[B_7(1 + a_3) + D_8 a_5]a_2 \quad \text{for } w_1 \leq 0, \end{aligned} \tag{A.4}$$

where

$$A_7 = C_{3,4}\frac{2^6}{13^3}\sqrt{3\pi}\sigma^2,$$

$$\begin{aligned} B_7 = \sqrt{78\pi}(&5^4 \cdot 7^4 w_1^{12} + 2^2 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 47 \cdot 67\sigma^2 w_1^{10} - 2 \cdot 3^4 \cdot 13^2 \cdot 19 \cdot 264221\sigma^4 w_1^8 \\ &+ 2^2 \cdot 3^4 \cdot 5 \cdot 13^3 \cdot 349 \cdot 15359\sigma^6 w_1^6 - 3^5 \cdot 5 \cdot 7 \cdot 13^4 \cdot 3161519\sigma^8 w_1^4 \\ &+ 2^3 \cdot 3^{11} \cdot 5 \cdot 7 \cdot 13^5 \cdot 1009\sigma^{10} w_1^2 + 2^4 \cdot 3^{13} \cdot 5 \cdot 7 \cdot 11 \cdot 13^6\sigma^{12}), \end{aligned}$$

$$\begin{aligned} D_7 = \sigma w_1(&-5^2 \cdot 7^4 w_1^{10} - 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 17\sigma^2 w_1^8 + 3^8 \cdot 13^2 \cdot 5231\sigma^4 w_1^6 - 3^4 \cdot 5 \cdot 7 \cdot 13^4 \cdot 17 \cdot 619\sigma^6 w_1^4 \\ &+ 2^3 \cdot 3^7 \cdot 7 \cdot 13^4 \cdot 26249\sigma^8 w_1^2 + 2^4 \cdot 3^8 \cdot 5 \cdot 7 \cdot 13^6 \cdot 47\sigma^{10}), \end{aligned}$$

$$D_8 = 2\cdot 3\cdot 7\cdot 13\sigma w_1 (5^4\cdot 7^2 w_1^{10} + 3\cdot 5^2\cdot 13^2\cdot 967\sigma^2 w_1^8 - 3^2\cdot 13^2\cdot 19\cdot 97397\sigma^4 w_1^6$$
$$+3^4\cdot 13^5\cdot 29\cdot 449\sigma^6 w_1^4 - 2^3\cdot 3^6\cdot 5\cdot 7\cdot 13^4\cdot 29\cdot 131\sigma^8 w_1^2 + 2^4\cdot 3^9\cdot 5\cdot 13^6\cdot 37\sigma^{10}),$$

$$C_{3,4} = \frac{1}{2^{14}\cdot 3^3\cdot 5\pi^{3/2}\sigma^{15}}.$$

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