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Finite-difference distributions for the Ginibre ensemble

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Abstract. The Ginibre ensemble of complex random matrices is studied. The complex-valued random variable of the second difference of complex energy levels is defined. For the $N = 3$ dimensional ensemble, we calculate distributions of the second difference real and imaginary parts, as well as its radius and of its argument (angle). For the generic N-dimensional Ginibre ensemble an exact analytical formula for the second difference's distribution is derived. Comparison with the real-valued random variable of the second difference of adjacent real-valued energy levels for a Gaussian orthogonal, unitary and symplectic ensemble of random matrices, as well as for a Poisson ensemble, is provided.

Keywords: Random matrix theory, Ginibre ensemble, finite-difference distributions

1. Introduction

Random matrix theory assumes that the Hamiltonian operator H of a generic quantum system is unknown and unknowable [1–3]. The matrix elements H_{ij} of the Hamiltonian in a given basis of Hilbert space are random variables. Their distributions are given by appropriate formulae depending on the studied random matrix ensemble [1–3]. The symmetry properties of H, which is Hermitian, lead us to Gaussian ensembles of random matrices: orthogonal GOE, unitary GUE and symplectic GSE, as well as to circular ensembles: orthogonal COE, unitary CUE and symplectic CSE. The energies E_i of quantum systems calculated from diagonalization of Hamiltonian matrix H_{ij} are random variables with appropriate distributions and they exhibit generic classes of level repulsion. It was Wigner who first discovered the level repulsion phenomenon [1–3]. The applications of random matrix theory are very broad: nuclear physics (slow neutron resonances, highly excited complex nuclei), condensed phase physics (fine metallic particles, random Ising model (spin glasses)), quantum chaos (quantum billiards, quantum dots) and disordered mesoscopic systems (transport phenomena). Ginibre studied the very general case of random Hamiltonians. He dropped the assumption of hermiticity of Hamiltonians and he considered generic complex-valued matrices [1, 2, 4, 5]. Thus, H belong to the general linear Lie group $\mathrm{GL}(N, \boldsymbol{C})$, where N is the dimension and $\boldsymbol{C}$ is the complex number field. Therefore, the energies Z_i of the quantum system ascribed to the Ginibre ensemble are complex valued. This is an extension of Gaussian or circular ensembles. Ginibre postulated the following joint probability density function of a random vector of complex eigenvalues $Z_1, \ldots, Z_N$ for $N \times N$ Hamiltonian matrices [1, 2, 4, 5]:

$$P(z_1, \ldots, z_N) = \prod_{j=1}^{N} \frac{1}{\pi \cdot j!} \cdot \prod_{i<j}^{N} |z_i - z_j|^2 \cdot \exp\left(-\sum_{j=1}^{N} |z_j|^2\right). \quad (1)$$

We emphasize that Z_i are *complex-valued* random variables, and z_i are *complex* sample points ($z_i \in \boldsymbol{C}$).

One must emphasize here the electrostatic analogy of Wigner and Dyson. A Coulomb gas of N unit charges moving on a complex plane (Gauss's plane) $\boldsymbol{C}$ is considered. The vectors of positions of charges are z_i and the potential energy of the system is

$$U(z_1, \ldots, z_N) = -\sum_{i<j} \ln|z_i - z_j| + \tfrac{1}{2}\sum_i |z_i^2|. \quad (2)$$

If the gas is in thermodynamical equilibrium at temperature $T = \frac{1}{2k_B}$ ($\beta = \frac{1}{k_B T} = 2$, k_B is Boltzmann's constant), then the probability density function of vectors of positions is $P(z_1, \ldots, z_N)$ (equation (1)). Thus, complex energies of a quantum system and vectors of positions of charges of a Coulomb gas are analogous to each other. In view of the above analogy one can consider the complex spacings $\Delta^1 Z_i$ of complex energies of a quantum system:

$$\Delta^1 Z_i = Z_{i+1} - Z_i, \qquad i = 1, \ldots, (N-1), \quad (3)$$

as vectors of relative positions of electric charges of a Coulomb gas. For the Ginibre ensemble the distributions of *real-valued* absolute values of spacings of nearest-neighbour *ordered* energies were calculated. We complement this by introduction of complex-valued second differences $\Delta^2 Z_i$ of complex energies:

$$\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, \qquad i = 1, \ldots, (N-2). \quad (4)$$

The second differences are three energy level magnitudes that enhance our knowledge of quantum systems with complex energies. Moreover, $\Delta^2 Z_i$ can be regarded as vectors of relative positions of vectors of relative positions of electric charges. One can observe movement of electric charges in the Cartesian frame of reference or in the polar one. Since the two-dimensional vectors in the Cartesian frame of reference have their projections on coordinate axes, then the real and imaginary parts of $\Delta^2 Z_i$, namely $\mathrm{Re}\,\Delta^2 Z_i$, $\mathrm{Im}\,\Delta^2 Z_i$, can be interpreted as projections of second differences on the abscissa and ordinate axes, respectively. The radii $|\Delta^2 Z_i|$, and arguments (angles) $\mathrm{Arg}\Delta^2 Z_i$ of the second differences have the interpretations of polar coordinates of $\Delta^2 Z_i$ vectors. $\Delta^2 Z_i$ are analogous to real-valued second differences:

$$\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, \qquad i = 1, \ldots, (N-2), \quad (5)$$

of adjacent ordered increasingly real-valued energies E_i defined for GOE, GUE, GSE and the Poisson ensemble PE (where the Poisson ensemble is composed of uncorrelated randomly distributed energies) [6–9]. We will calculate the distributions of $\Delta^2 Z_i$, $\mathrm{Re}\,\Delta^2 Z_i$, $\mathrm{Im}\,\Delta^2 Z_i$, $|\Delta^2 Z_i|$ and $\mathrm{Arg}\Delta^2 Z_i$. Finally, we will compare these results with second-difference distributions for Gaussian ensembles and the Poisson ensemble [1, 10–15].

2. Second-difference distributions

We use formula (1) with $N = 3$ and define the following complex-valued random vector (Y_1, Y_2, Y_3) and real A_j and imaginary B_j parts:

$$\begin{gathered} Y_1 = \Delta^2 Z_1, \qquad Y_2 = Z_2 - Z_3, \\ Y_3 = Z_3, \qquad Y_j = (A_j, B_j), \\ A_j = \mathrm{Re}\,Y_j, \qquad B_j = \mathrm{Im}\,Y_j, \qquad j = 1, \ldots, 3. \end{gathered} \quad (6)$$

The change of the variable formula gives us the result for the joint probability density function of random vector (Y_1, Y_2, Y_3) [16]:

$$\begin{aligned} &f_{(Y_1,Y_2,Y_3)}(y_1, y_2, y_3) \\ &\quad = f_{(A_1,B_1,A_2,B_2,A_3,B_3)}(a_1, b_1, a_2, b_2, a_3, b_3) \\ &\quad = \frac{1}{12\pi^3} \cdot [(a_1 + a_2)^2 + (b_1 + b_2)^2] \cdot [a_2^2 + b_2^2] \\ &\qquad \times [(a_1 + 2a_2)^2 + (b_1 + 2b_2)^2] \\ &\qquad \times [\exp(-(a_1 + 2a_2 + a_3)^2 - (b_1 + 2b_2 + b_3)^2 \\ &\qquad -(a_2 + a_3)^2 - (b_2 + b_3)^2 - a_3^2 - b_3^2)], \end{aligned} \quad (7)$$

where $y_j = (a_j, b_j) \in \boldsymbol{C}$ are complex random sample points. (In order to obtain equation (7) we used the complex-valued linear map

$$(Y_1, Y_2, Y_3) = \Xi(Z_1, Z_2, Z_3), \qquad \Xi = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

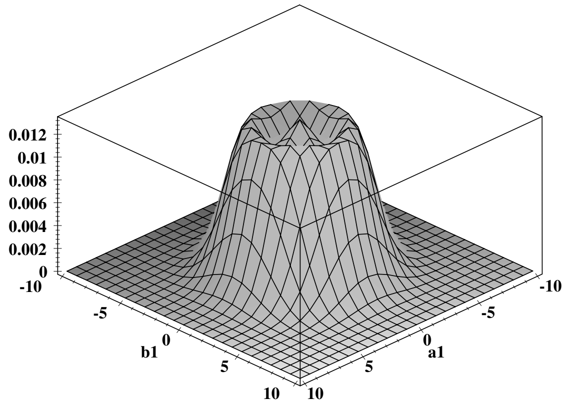

Figure 1. The probability density function of the complex-valued second difference Y_1 for the Ginibre ensemble.

where the Jacobian of the inverse map Ξ^{-1} is equal to unity $\mathrm{Jac}(\Xi^{-1}) = 1$). We integrate out Y_3:

$$f_{(Y_1,Y_2)}(y_1, y_2) = \int_C f_{(Y_1,Y_2,Y_3)}(y_1, y_2, y_3)\,\mathrm{d}y_3, \quad (9)$$

and we obtain the following marginal probability density function:

$$\begin{aligned} &f_{(Y_1,Y_2)}(y_1, y_2) = f_{(A_1,B_1,A_2,B_2)}(a_1, b_1, a_2, b_2) \\ &\quad = \frac{1}{36\pi^2}[(a_1 + a_2)^2 \\ &\qquad +(b_1 + b_2)^2][a_2^2 + b_2^2][(a_1 + 2a_2)^2 + (b_1 + 2b_2)^2] \\ &\qquad \times \exp[-\tfrac{2}{3}a_1^2 - 2a_1a_2 - 2a_2^2 - \tfrac{2}{3}b_1^2 - 2b_1b_2 - 2b_2^2]. \end{aligned} \quad (10)$$

Now we calculate the marginal probability density function of the second difference:

$$\begin{aligned} f_{Y_1}(y_1) &= f_{(A_1,B_1)}(a_1, b_1) \\ &= \int_C f_{(Y_1,Y_2)}(y_1, y_2)\,\mathrm{d}y_2 \\ &= \frac{1}{576\pi}[(a_1^2 + b_1^2)^2 + 24] \cdot \exp\left(-\frac{1}{6}(a_1^2 + a_2^2)\right). \end{aligned} \quad (11)$$

We plot the distribution of Y_1 in figure 1. Now we derive the distributions of the real part A_1 and of the imaginary part B_1 of the second difference:

$$\begin{aligned} f_{A_1}(a_1) &= \int_R f_{(A_1,B_1)}(a_1, b_1)\,\mathrm{d}b_1 \\ &= \frac{\sqrt{6}}{576\sqrt{\pi}}(a_1^4 + 6a_1^2 + 51) \cdot \exp\left(-\frac{1}{6}a_1^2\right), \end{aligned} \quad (12)$$

$$\begin{aligned} f_{B_1}(b_1) &= \int_R f_{(A_1,B_1)}(a_1, b_1)\,\mathrm{d}a_1 \\ &= \frac{\sqrt{6}}{576\sqrt{\pi}}(b_1^4 + 6b_1^2 + 51) \cdot \exp\left(-\frac{1}{6}b_1^2\right), \end{aligned} \quad (13)$$

where $\boldsymbol{R}$ is the field of real numbers. We plot the real part's distribution in figure 2.

We transform the complex-valued random variable of the second difference Y_1 to polar coordinate variables R_1, Φ_1:

$$R_1 = |Y_1|, \qquad \Phi_1 = \mathrm{Arg}Y_1, \quad (14)$$

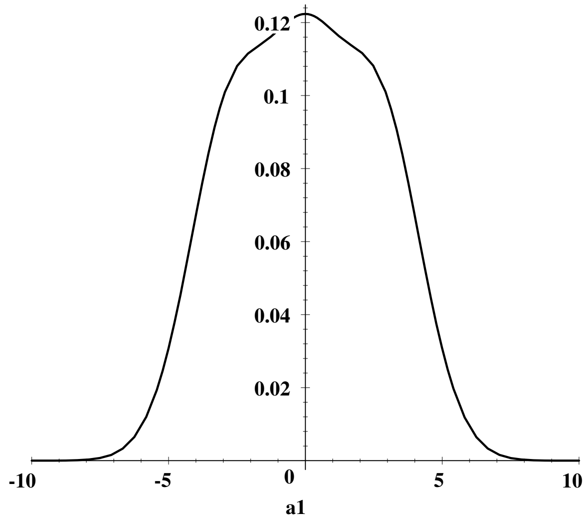

Figure 2. The probability density function of the real part A_1 of the second difference for the Ginibre ensemble.

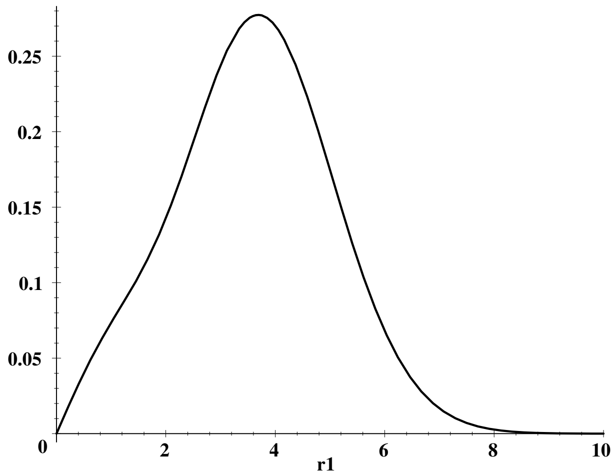

Figure 3. The probability density function of the radius R_1 of the second difference for the Ginibre ensemble.

and we obtain by the standard method the following probability density function of random vector (R_1, Φ_1):

$$f_{(R_1,\Phi_1)}(r_1, \phi_1) = \Theta(r_1)\frac{1}{576\pi}r_1(r_1^4 + 24) \cdot \exp\left(-\frac{1}{6}r_1^2\right) \tag{15}$$

(the Jacobian of the transformation is r_1, Θ, which is the Heaviside (step) function [16]). It follows that

$$f_{R_1}(r_1) = \Theta(r_1)\tfrac{1}{288}r_1(r_1^4 + 24) \cdot \exp(-\tfrac{1}{6}r_1^2),$$
$$f_{\Phi_1}(\phi_1) = \frac{1}{2\pi}, \qquad \phi_1 \in [0, 2\pi]. \tag{16}$$

We present the distribution of R_1 in figure 3.

3. The *N*-dimensional Ginibre ensemble

The case of the generic N-dimensional Ginibre ensemble is of special physical interest ($N \geqslant 3$). We will calculate the distribution of the second difference for the ensemble. One commences with the n-level correlation function [1]:

$$P_n(z_1, \ldots, z_n) = \int_{C^{N-n}} P(z_1, \ldots, z_N)\,\mathrm{d}z_{n+1} \ldots \mathrm{d}z_N \tag{17}$$

$$= \pi^{-n} \exp\left(-\sum_{i=1}^{n} |z_i|^2\right) \det D^{(n)}, \tag{18}$$

$$D_{ij}^{(n)} = e_{N-1}(z_i z_j^*), \qquad i, j = 1, \ldots, n,$$
$$e_{N-1}(z) = \sum_{k=0}^{N-1} \frac{z^k}{k!}. \tag{19}$$

In order to calculate the distribution of the complex second difference $W_1 = \Delta^2 Z_1$ for the N-dimensional Ginibre ensemble one substitutes $n = 3$ into equation (17) and defines random vector $W = (W_1, W_2, W_3)$:

$$(W_1, W_2, W_3) = \Omega(Z_1, Z_2, Z_3),$$
$$\Omega = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

The probability density function of random vector W reads [16]

$$P_3(w_1, w_2, w_3) = P_3(\Omega^{-1}(w_1, w_2, w_3))$$
$$= P_3(w_1 + 2w_2 - w_3, w_2, w_3). \tag{21}$$

Hence, the distribution of the second difference is

$$P_3(w_1) = \int_{C^2} P_3(w_1 + 2w_2 - w_3, w_2, w_3)\,\mathrm{d}w_2\,\mathrm{d}w_3. \tag{22}$$

We combine equations (17)–(19), (21) and (22), and we use Laplace's expansion of the determinant $\det D^{(n)}$:

$$P_3(w_1) = \pi^{-3} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \int_{C^2} \exp\left(-\sum_{i=1}^{3} |(\Omega^{-1}w)_i|^2\right)$$
$$\times \prod_{k=1}^{3} e_{N-1}[(\Omega^{-1}w)_k \cdot (\Omega^{-1}w)^*_{\mathcal{P}k}]\,\mathrm{d}w_2\,\mathrm{d}w_3, \tag{23}$$

where $\mathcal{P}$ is the permutation of indices $(1, 2, 3)$. The only nonzero contribution to equation (23) is for the identity permutation $\mathcal{P} = \mathrm{id} = (1, 2, 3)$. This results from the fact that other permutations produce factors that are periodic functions of arguments $\mathrm{Arg}w_2$, $\mathrm{Arg}w_3$ of complex numbers w_2, w_3 (the integrals over w_2, w_3 can be transformed to polar coordinates where the arguments $\mathrm{Arg}w_2$, $\mathrm{Arg}w_3$ are integrated over $[0, 2\pi]$). Hence,

$$P_3(w_1) = \pi^{-3} \int_{C^2} \exp\left(-\sum_{i=1}^{3} |(\Omega^{-1}w)_i|^2\right)$$
$$\times \prod_{k=1}^{3} e_{N-1}[|(\Omega^{-1}w)_k|^2]\,\mathrm{d}w_2\,\mathrm{d}w_3. \tag{24}$$

Therefore, considering equation (19), one has

$$P_3(w_1) = \pi^{-3} \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} \frac{1}{j_1!j_2!j_3!} I_{j_1 j_2 j_3}(w_1), \tag{25}$$

$$I_{j_1 j_2 j_3}(w_1) = \int_{C^2} \exp\left(-\sum_{i=1}^{3} |(\Omega^{-1}w)_i|^2\right)$$
$$\times \prod_{k=1}^{3} |(\Omega^{-1}w)_k|^{2j_k}\,\mathrm{d}w_2\,\mathrm{d}w_3. \tag{26}$$

One changes variables in equation (26) in the following way: $V_2 = 2W_2$, $V_3 = -W_3$, and obtains

$$I_{j_1 j_2 j_3}(w_1) = 2^{-2j_2} \int_{C^2} \exp(-|w_1 + v_2 + v_3|^2 - \tfrac{1}{4}|v_2|^2 - |v_3|^2)|w_1 + v_2 + v_3|^{2j_1}|v_2|^{2j_2}|v_3|^{2j_3}\, \mathrm{d}v_2\, \mathrm{d}v_3. \quad (27)$$

The above double integral can be calculated by extending the exponent by additional terms proportional to λ_i parameters and considering appropriate derivatives:

$$I_{j_1 j_2 j_3}(w_1) = 2^{-2j_2} \frac{\partial^{j_1+j_2+j_3}}{\partial^{j_1}\lambda_1 \partial^{j_2}\lambda_2 \partial^{j_3}\lambda_3} \times F(w_1, \lambda_1, \lambda_2, \lambda_3)|_{\lambda_i=0}, \quad (28)$$

$$F(w_1, \lambda_1, \lambda_2, \lambda_3) = \int_{C^2} \times \exp[G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3)]\, \mathrm{d}v_2\, \mathrm{d}v_3, \quad (29)$$

$$G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3) = (\lambda_1 - 1)|w_1 + v_2 + v_3|^2 + (\lambda_2 - \tfrac{1}{4})|v_2|^2 + (\lambda_3 - 1)|v_3|^2. \quad (30)$$

Finally, we derive $F(w_1, \lambda_1, \lambda_2, \lambda_3)$ by transformation of the parametric quadratic form $G(w_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3)$ to the canonical form and integrating over v_2, v_3:

$$F(w_1, \lambda_1, \lambda_2, \lambda_3) = A(\lambda_1, \lambda_2, \lambda_3) \times \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2], \quad (31)$$

where

$$A(\lambda_1, \lambda_2, \lambda_3) = \frac{(2\pi)^2}{(\lambda_1 + \lambda_2 - \frac{5}{4}) \cdot (\lambda_1 + \lambda_3 - \frac{5}{4}) - (\lambda_1 - 1)^2}, \quad (32)$$

$$B(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 - 1) \cdot \frac{2\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{2}}{2\lambda_1 + \lambda_2 + \lambda_3 - \frac{9}{2}}. \quad (33)$$

Hence, we obtained the analytical formula for the distribution $P_3(w_1)$ of the second difference for the N-dimensional Ginibre ensemble combining equations (25), (28) and (31)–(33). It is worth mentioning that the second difference's distribution is a triple sum of zero-centred Gaussian distributions with different widths. The distribution is again a function of only modulus $|w_1|$ of the second difference and it has a global maximum at the origin.

4. Comparison

Finally, in order to compare our results for the second difference for the Ginibre ensemble with previous ones for Gaussian and Poissonian ensembles we must rescale them by division by appropriate magnitude. We consider such rescaled dimensionless second differences in the following way. The mean values of second differences either in the real or in the complex case are zero; hence we cannot divide the second differences by the mean values. It follows that we divide real-valued second differences $\Delta^2 E_1$ for GOE(3) ($\beta = 1$), for GUE(3) ($\beta = 2$), for GSE(3) ($\beta = 4$) and for PE ($\beta = 0$) by mean spacings $\langle S_\beta \rangle$ calculated for these ensembles, and we create new dimensionless second differences:

$$C_\beta = \frac{\Delta^2 E_1}{\langle S_\beta \rangle}, \quad (34)$$

respectively [6–9]. The probability density functions of C_β were calculated for GOE, GUE, GSE and PE [6–9]. Since the second difference for the Ginibre ensemble is complex valued, we choose its real part A_1 for comparison with C_β. One divides A_1 by the analogue of $\langle S_\beta \rangle$, which is the mean value $\langle R_1 \rangle$ of the radius R_1 equation (16) of $\Delta^2 Z_1$:

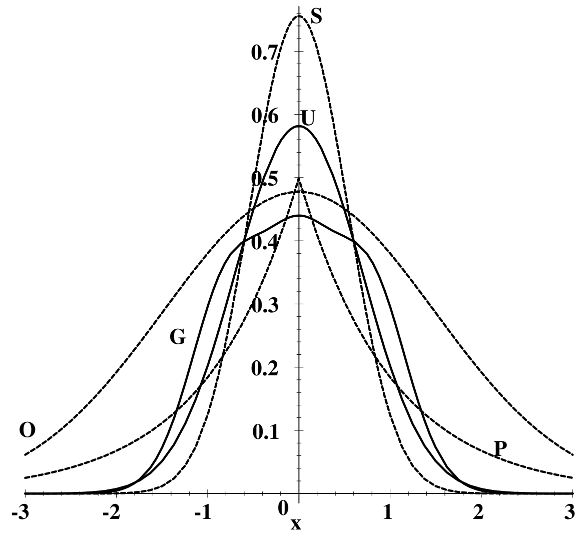

Figure 4. The probability density function of the rescaled second differences for the Ginibre ensemble X_1 (solid curve: G), for GOE(3) C_1 (dashed curve: O), for GUE(3) C_2 (solid curve: U), for GSE(3) C_4 (dashed curve: S) and for PE C_0 (dashed curve: P), respectively.

$$\langle R_1 \rangle = \int_0^\infty r_1 f_{R_1}(r_1)\, \mathrm{d}r_1 = \tfrac{53}{64}\sqrt{6\pi}. \quad (35)$$

Hence

$$X_1 = \frac{A_1}{\langle R_1 \rangle}, \quad (36)$$

is the rescaled dimensionless A_1. The probability distributions of C_β and of X_1 depend on the same real variable x, which is equal to $\frac{\Delta^2 e_1}{\langle S_\beta \rangle}$, $\frac{a_1}{\langle R_1 \rangle}$, respectively ($e_1$ is the value of energy E_1). We plot in figure 4 the probability density functions of rescaled second differences X_1, C_β, for the Ginibre ensemble, GOE(3), GUE(3), GSE(3) and PE, respectively. One can infer from figure 4 that the second differences' distributions for Gaussian, Poisson and Ginibre ensembles assume global maxima at origin and that they are unimodular. Firstly, it extends the theorem of level homogenization to the Ginibre ensemble [6–9]. We can formulate the following law: *energy levels for Gaussian ensembles, for the Poisson ensemble and for the Ginibre ensemble tend to be homogeneously distributed.* The second differences' distributions assume global maxima at the origin no matter whether the second differences are real or complex. From the Coulomb gas point of view this is easier to understand. The unit charges behave in such a way that the vectors of relative positions of vectors of relative positions of charges statistically tend to be zero. This could be called stabilization of the structure of the system of electric charges. The above results can be extended to the study of higher differences' distributions for the Ginibre ensemble.

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